Indian Institute of Technology Ropar
Department of Mathematics
MA101 - Calculus
First Semester of Academic Year 2025-26

Tutorial Sheet - 6

1. Compute $L(P, f)$ and $U(P, f)$, if $f(x) = x^2$ on [0,1] and $P = \left\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\right\}$ be a partition of [0,1].

2. Let f be a bounded function defined on $[a, b]$. M and m are such that $m = inf_{x\in[a,b]}f(x)$, $M = sup_{x\in[a,b]}f(x)$. Prove that:

   (a) Let P and Q be two partitions of $[a, b]$ such that $P \subset Q$. Then $L(P, f) \le L(Q, f) \le U(Q, f) \le U(P, f)$.

   (b) Let P and Q be arbitrary partition of $[a, b]$. Then $L(P, f) \le U(Q, f)$.

   (c) Let P be a partition of $[a, b]$. If Q is a refinement of P, then $U(P, f) - L(P, f) \ge U(Q, f) - L(Q, f)$.

3. Let $f : [0, 1] \to [0, 1]$ be defined by

$$f(x) = \begin{cases} \dfrac{1}{q}, & x = \dfrac{p}{q}, (p, q) = 1 \\ 0, & otherwise \end{cases}$$

   and $g : [0, 1] \to [0, 1]$ be defined by

$$g(x) = \begin{cases} 0, & x = 0 \\ 1, & otherwise \end{cases}$$

   Show that both f and g are integrable. Also check whether g o f is integrable.

4. If $f(x) = x^3$ is defined on $[0, a]$. Show that f is Riemann integrable over $[0, a]$ and $\displaystyle\int_0^a f(x)dx = \frac{a^4}{4}$.(Use Riemann theory)

5. If f is non-negative continuous function on $[a, b]$ and $\displaystyle\int_a^b f(x)dx = 0$. Then $f(x) = 0\ \forall x \in [a, b]$.

6. Evaluate the limit $\displaystyle\lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n}\right)$ as an integral.

7. Evaluate the limit $\displaystyle\lim_{n\to\infty}\left[\left(1 + \frac{1^2}{n^2}\right)\left(1 + \frac{2^2}{n^2}\right)\ldots\left(1 + \frac{n^2}{n^2}\right)\right]^{\frac{1}{n}}$ as an integral.

8. Let $f(x) = x^5 \log(1 + x)$ for $x > -1$. For any $\epsilon > 0$, find a polynomial $P_n(x)$ such that

$$\int_0^1 P_n(x)dx - \epsilon < \int_0^1 f(x)dx < \int_0^1 P_n(x)dx + \epsilon.$$

9. Give a counterexample to show that the converse of the fundamental theorem of calculus(FTC) is not true.

10. Prove the following inequalities using MVT for integrals.

(a) $\frac{1}{10\sqrt{2}} \leq \int_0^1 \frac{x^9 dx}{\sqrt{1+x}} \leq \frac{1}{10}$ (b) $0 \leq \int_0^{\frac{\pi}{2}} e^x \sin x dx \leq e^{\frac{\pi}{2}} - 1$.

11. Use upper and lower Riemann sum for the function $f(x) = \frac{1}{x}$ and find the upper and lower bound for the sum $\sum_{i=1}^{n} \frac{1}{i}$.

12. Check whether FTC can be used to evaluate the followings:

(a) $\frac{d}{dx}\left(\int_x^{x^2} e^{-t^2} dt\right)$

(b) $\frac{d}{dx}\left(\int_0^{\sin x} \sqrt{t^2+1} dt\right)$.

If **yes**, then evaluate or if **no**, then justify your answer.

13. Let $f(x)$ be defined on $[0, 2]$ such that

$$f(x) = \begin{cases} x + x^2, & x \in \mathbb{Q} \\ x^2 + x^3, & x \in \mathbb{Q}^c \end{cases}$$

Find the upper and lower Riemann integral of f over $[0, 2]$. Is f Riemann integrable over $[0, 2]$?

14. Let $g(x)$ be defined on $[0, 1]$ such that

$$f(x) = \begin{cases} (1 - x^2)^{\frac{1}{2}}, & x \in \mathbb{Q} \\ (1 - x), & x \in \mathbb{Q}^c \end{cases}$$

Find the upper and lower Riemann integral of g over $[0, 1]$. Is g Riemann integrable over $[0, 1]$?

15. Show that f and g are not integrable on $[a, b]$ but fg(product of two functions) is integrable on $[a, b]$ where

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \in \mathbb{Q}^c \end{cases}, \quad g(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{Q}^c \end{cases}$$

16. (a)Show that the function f defined on $[0, 4]$ by $f(x) = [x]$ is integrable on $[0, 4]$ and $\int_0^4 f(x)dx = 6$.

(b)Evaluate $\int_0^2 x[2x]dx$, where $[x]$ is the greatest integer $\leq x$.

17. Prove that every monotone function on $[a, b]$ is integrable. Is the converse true?

18. For each $x \in [0, 1]$, $x = 0.x_1x_2x_3\ldots$ be the decimal expansion of x, not eventually all 9s. Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = x_1$. Evaluate $\int_0^1 f(x)dx$.

19. (a) Using change of variables, evaluate $\int_0^3 \frac{t}{\sqrt{1+t^2}}dt$

(b) For $n \geq 1$, verify that $\int_{-\pi}^{\pi} |x| \cos nx dx = \begin{cases} 0, & if \quad n \ is \ even \\ \frac{-4}{n^2}, & if \quad n \ is \ odd \end{cases}$

20. Verify the limit $\lim_{n\to\infty} \left(\frac{n^n}{n!}\right)^{\frac{1}{n}} = e$ using integral.

21. Disprove the statement with example $\int_a^b f'(x)dx = f(b) - f(a)$.

22. Find the value of c that satisfy the MVT for integral on $[\frac{3\pi}{4}, \pi]$ for $f(x) = \cos(2x - \pi)$.

23. Determine the number c that satisfies the MVT for integrals on $[1, 4]$ for $f(x) = x^2 + 3x + 2$.

24. Find the values of c that satisfy the MVT for integrals on $[0, 1]$ where $f(x) = x(1 - x)$.

25. Find the first 3 terms in the Maclaurin series for (a)xe^{-x} (b)$\sqrt{1 - x + x^2}$.

26. Show that $\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots -1 < x \leq 1$ and express $\frac{\pi}{2}$ in series.

27. Evaluate the integral $\int_0^1 e^{-x^2} dx$ using Taylor series.

28. Find the Maclaurin series for the function $e^{\sin x}$ upto the term in x^4 and evaluate $\int_0^1 e^{\sin x} dx$.

29. Evaluate the integral $\int_0^{\frac{\pi}{6}} \sin^2 x dx$ by finding the Maclaurin approximation to the integrand with 3 terms in the series(upto 4 decimals).

30. Find the first 3 terms in the Maclaurin series for $\cos(\sin x)$ and evaluate $\lim_{x\to 0} \frac{1 - \cos(\sin x)}{x^2}$.

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