Indian Institute of Technology Ropar
Department of Mathematics
MA101 - Calculus
First Semester of Academic Year 2025-26

Tutorial Sheet - 5

1. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfied the condition $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. If f is differentiable at 0, prove that f is differentiable at every $c \in \mathbb{R}$ and $f^{'}(c) = f^{'}(0)f(c)$.

2. A function f is defined on some neighborhood of c and f is differentiable at c. Prove that
$$\lim_{h \to 0} \frac{f(c+h) - f(c-h)}{2h} = f^{'}(c).$$

3. A function f is defined on $\mathbb{R}$ by $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$. Show that f is differentiable at 0 but $f^{'}$ is not continuous at 0.

4. Prove taht the function $f(x) = \begin{cases} x^r \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ is

   (a) continuous from the right at $0 \Leftrightarrow r > 0$. (b) differentiable from the right at $0 \Leftrightarrow r > 1$.

5. Prove that if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is an even function [(i.e.) $f(-x) = f(x)$ for all $x \in \mathbb{R}$] and has a derivative at every point, then the derivative $f^{'}$ is an odd function [(i.e.) $f^{'}(-x) = -f^{'}(x)$ for all $x \in \mathbb{R}$]. Also prove that if $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a differentiable odd function, the g is an even function.

6. Find $\frac{dz}{dt}$, where $z = f(x, y) = x^5 y^6$, $x(t) = e^t$ and $y(t) = \sqrt{t}$.

7. Find $\frac{dw}{dt}$, where $w = f(x, y, z) = 2y - \sin(xz)$, $x(t) = 3t$, $y(t) = e^{t-1}$ and $z(t) = \ln t$.

8. Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a)$ and $f(b)$ are of different signs and $f^{'}(x) \neq 0$ for all $x \in (a, b)$, show that there is a unique $x_0 \in (a, b)$ such that $f(x_0) = 0$.

9. Using the MVT, prove that

   (a)$|\sin a - \sin b| \leq |a - b|$ for all $a, b \in \mathbb{R}$. (b)$\frac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \frac{v-u}{1+u^2}$, if $0 < u < v$.

10. Let $a > 0$ and f be continuous on $[-a, a]$. Suppose that $f^{'}(x)$ exists and $f^{'}(x) \leq 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, show that $f(0) = 0$.

11. Suppose we have 50 mango trees in our campus and each produces 800 fruits each year. Our gardener wants to plant mango trees in our campus, but each additional plants drops down the production by 10 for each existing trees. How many number of trees requires to plant to get maximum mango production? Also find maximum production.

12. Two corridors meet perpendicularly at a corner. One of the corridors is of 2 meter wide and the other one is of 3 meter wide. What is the maximum length of the iron rod required on the floor that passes through the point P and touches both the walls 1 and 2?(see Figure 1).

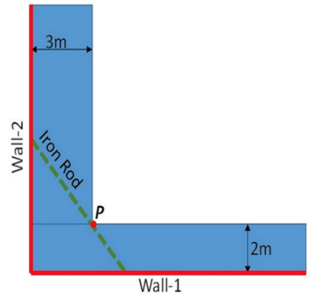

Figure 1.

13. Determine where the function $f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}$ is differentiable and find its derivative.

14. Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that $|f(x) - f(y)| \leq |x - y|^2$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function.

15. Find the domain of continuity and differentiability of the following functions and find its derivative.
(a)$f(x) = |\sin x|$ (b)$g(x) = x - [x]$ (c)$h(x) = 2x + |x + 1|$.

16. Find the points of relative(local) extrema of the following functions on the specified domain.
(a)$f(x) = x|x^2 - 12|$ for $-2 \leq x \leq 3$ (b)$g(x) = 1 - (x - 1)^{\frac{2}{3}}$ for $0 \leq x \leq 2$.

17. Let $f(x) = \begin{cases} x^2, & x \geq 0 \\ 0, & x < 0 \end{cases}$ (a)Prove that f is differentiable at 0.
(b)Is f' continuous at 0?(prove or disprove) (c) Is f' differentiable at 0?(prove or disprove)

18. Let $f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{Q}^c \end{cases}$. Prove that f is differentiable at 0 and find $f'(0)$. Is f differentiable anywhere else? Explain.

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