Indian Institute of Technology Ropar
Department of Mathematics
MA101 - Calculus
First Semester of Academic Year 2025-26

Tutorial Sheet - 4

1. Given $f(x) = \dfrac{x^2+6x+5}{x+5}$, $x_0 = -5$, $\epsilon = 0.03$. Find $L = \lim_{x\to x_0} f(x)$. Then find a number $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta \Longrightarrow |f(x) - L| < \epsilon$.

2. Evaluate the limits of the following, if they exist:

   (a) $\lim_{x\to 0} \dfrac{\cos(2x-1)}{\cos(x-1)}$ (b) $\lim_{x\to -2} \dfrac{\frac{1}{x}-\frac{1}{2}}{x^3-8}$ (c) $\lim_{x\to \frac{\pi}{2}} \dfrac{\tan 2x}{x-\frac{\pi}{2}}$

   (d) $\lim_{x\to 0} \dfrac{\sqrt{x^2+4}-2}{x^2}$ (e) $\lim_{x\to 1} \dfrac{x^m-1}{x^n-1}, m, n \in \mathbb{N}$ (f) $\lim_{x\to 0} \dfrac{\sin 3x \cot 5x}{x \cot 4x}$

3. Find $\lim_{x\to 0^-} \dfrac{|x|}{x}$ and $\lim_{x\to 0^+} \dfrac{|x|}{x}$

4. Prove that $\lim_{x\to 0} f(x) = 0$, if $f(x) = 2x$ for $x < 0$ and $f(x) = \dfrac{x}{2}$ for $x \geq 0$.

5. Prove that $\lim_{x\to 0} \cos\dfrac{1}{x}$ does not exist but $\lim_{x\to 0} x\cos\dfrac{1}{x} = 0$.

6. Suppose the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ has limit L at 0 and let $a > 0$. If $g : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $g(x) = f(ax)$ for $x \in \mathbb{R}$, show that $\lim_{x\to 0} g(x) = L$.

7. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \in \mathbb{Q}^c \end{cases}$$

   Using $\epsilon - \delta$ definition of limit, prove that limit of $f(x)$ does not exist at any point in $\mathbb{R}$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{x\to\alpha} f(x)$ exists for $\alpha \in \mathbb{R}$. Show that

$$\lim_{h\to 0} [f(\alpha+h) - f(\alpha-h)] = 0.$$

   Analyze the converse.

9. Consider the limit statement: $\lim_{x\to 2} x^2 = 4$. Find a value of $\delta > 0$ that will guarantee that whenever x is within distance δ from 2(but not equal to 2), x^2 is within distance 0.01 from 4.

10. Investigate left and right hand limits for the following functions as $x \to x_0$ and determine whether limit exists.
    (a) $f(x) = x[x]$, $x_0 = 0$ (b) $g(x) = x^2[x]$, $x_0 = 1$ (c) $h(x) = (-1)^{[x]-[x^2]}$, $x_0 = 2$
    where $[x]$ is greatest integer $\leq x$.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 4x^2 + 2x - 11 & \text{if } x \in \mathbb{Q} \\ 3x^2 + x - 5 & \text{if } x \in \mathbb{Q}^c \end{cases}$ Show that $\lim_{x \to \alpha} f(x)$ exist for precisely two values of α.

12. Let $c \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{x \to c} [f(x)]^2 = L$.
(a) Show that if $L = 0$, then $\lim_{x \to c} f(x) = 0$.
(b) Show by example if $L \neq 0$ then f may not have limit at c.

13. Determine whether the following limit exists in $\mathbb{R}$:

$$\lim_{x \to 0} sgn \left[\sin \left(\frac{1}{x} \right) \right], \ x \neq 0.$$

where $sgn(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$

14. Discuss the continuity of the following functions:
(a) $f(x) = \sin \frac{1}{x}$ if $x \neq 0$ and $f(0) = 0$

(b) $f(x) = \begin{cases} \dfrac{x}{[x]} & \text{if } 1 \leq x < 2 \\ 1 & \text{if } x = 2 \\ \sqrt{6 - x} & \text{if } 2 \leq x \leq 3 \end{cases}$

15. Using $\epsilon - \delta$ definition, prove the continuity of the following functions:
(a) $f(x) = \cos x$ for all $x \in \mathbb{R}$.
(b) $f(x) = x^2 \cos \left(\frac{1}{x} \right)$ if $x \neq 0$ and $f(0) = 0$
(c) $f(x) = x \sin \left(\frac{1}{x} \right)$ if $x \neq 0$ and $f(0) = 0$

16. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at $x = 0$, show that f is continuous at every $x \in \mathbb{R}$.

17. Let $g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 1 - x & \text{if } x \in \mathbb{Q}^c \end{cases}$. Show that g is continuous only at $x = \frac{1}{2}$.

18. If $f : [0, 1] \to \mathbb{R}$ is continuous and has only rational[respectively, irrational] values, then f must be constant. Prove your assertion.

19. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and $f(r) = 0$ for every rational number r. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

20. (a) Suppose that the function $f : [0, 1] \to \mathbb{R}$ is continuous and that $f(x) \geq 2$ if $0 \leq x < 1$. Then show that $f(1) \geq 2$.
(b) Suppose f is continuous on [0,1] and $f(x) > 2$ if $0 \leq x < 1$. Is it necessary the case that $f(1) > 2$?

21. Determine the intervals over which the following functions are continuous.
(a) $f(x) = \sin \left(\frac{x + 2}{x - 2} \right)$ (b) $g(x) = \cos(\sqrt{x})$ (c) $h(x) = \tan(\sin(x))$

22. If f, g are continuous at $x_0 \in \mathbb{R}$, then show that $max\{f, g\}$ and $min\{f, g\}$ are continuous at x_0.

23. Let $f : [0, 1] \to \mathbb{R}$ have the property that the limit $g(x) := \lim_{t \to x} f(t)$ exists in $\mathbb{R}$ for all $x \in [0, 1]$, then show that g is continuous.

24. Give an example of a function $f : [0, 1] \to \mathbb{R}$ that is discontinuous at every point of [0,1] but $|f|$ is continuous on [0,1].

25. Determine the points of discontinuity of the function:
(a) $g(x) = \dfrac{1}{2 - 4\cos(3x)}$ (b) $h(x) = \dfrac{e^{x^2+1}}{e^x - 2e^{1-x}}$.

26. Consider the function $f(x) = \begin{cases} \log_e x & \text{if } 0 < x < 1 \\ ax^2 + b & \text{if } 1 \leq x < \infty \end{cases}$, if $f(2) = 3$, determine the values of a and b for which $f(x)$ is continuous.

27. Let $f(x) = \begin{cases} \dfrac{1}{(x+3)^2} & \text{if } x \leq -1 \\ 2 - x & \text{if } -1 < x \leq 1 \\ \dfrac{3}{x+2} & \text{if } x > 1 \end{cases}$, find all values of x where f is NOT continuous.

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