Indian Institute of Technology Ropar
Department of Mathematics
MA101 - Calculus
First Semester of Academic Year 2025-26

Tutorial Sheet - 2

1. For the given sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, prove or disprove the following:

   (i) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent.
   (ii) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent and $\{b_n\}_{n\geq 1}$ is bounded.

2. Show that the sequence $\left\{\dfrac{n^2+3n+5}{2n^2+5n+7}\right\}$ converges to $\dfrac{1}{2}$.

3. Test the convergence/divergence of the following sequences by using the definition of convergence/divergence.

   (i) $\dfrac{n}{n+1}$ (ii) $\dfrac{2n}{3n^2+1}$ (iii) $\dfrac{2n^2+3}{3n^2+1}$

4. Test the convergence/divergence of the following sequences by using algebra of limits for sequences.

   (i) $x_n = \dfrac{P(n+1)}{P(n)}$ where $P(t) = 5t^4 + t^2 + 1$

   (ii) For $0 < a < b$, consider $x_n = \dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$.

5. Prove that $\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0$, if $p > 0$.

6. Determine whether the following sequences converges/diverges.
   (i) $\sqrt{n+1} - \sqrt{n}$ (ii) n^{1/n^2} (iii) $(n!)^{1/n^2}$

7. For $a > 0$, let $S_1 = a$ and $S_{n+1} = \dfrac{1}{2}\left(S_n + \dfrac{a}{S_n}\right)$. Prove that $\{S_n\}$ converges to $\sqrt{a}$.

8. Prove that the following sequences converges and find its limits.
   (i) $a_1 = 1$ and $a_{n+1} = \sqrt{2 + a_n}$ for all $n \geq 1$.
   (ii) $a_1 = 1$ and $a_{n+1} = \dfrac{1}{4}(2a_n + 3)$ for all $n \geq 1$.

9. If $x_n > 0$ for all $n \in \mathbb{N}$, then prove that $x_n \to \infty$ as $n \to \infty$ if and only if $\dfrac{1}{x_n} \to 0$ as $n \to \infty$.

10. If $a_n \to a$, $b_n \to b$ and $a_n \leq b_n$ $\forall n \in \mathbb{N}$, then prove that $a \leq b$.

11. Prove that the sequence $\{x_n\}$, where $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n}$ is convergent.

12. Prove that the sequence $\{1 + (-1)^n\}$ oscillates finitely.

13. Prove that the following sequences are convergent by showing that they are monotone and bounded.

(i) $x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}$, for $n \geq 1$.

(ii) $x_1 > 0$ and $x_{n+1} = \frac{1}{2}\left(x_n + \frac{9}{x_n}\right)$, for $n \geq 1$.

14. Prove that the sequence $\{x_n\}$ is divergent, where $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$.

15. Using Squeeze theorem, prove that

(i) $\lim_{n\to\infty}\left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \ldots + \frac{1}{(n+n)^2}\right) = 0.$

(ii) $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots + \frac{1}{\sqrt{n^2+n}}\right) = 1.$

******* End *******