Tutorial Sheet - 13

1. Verify both forms of Green's theorem for the field $\vec{F} = M\hat{i} + N\hat{j}$. Take the domains of integration in each case to be the disk $R : x^2 + y^2 \le a^2$ and its bounding circle $C : \vec{r} = acos(t)\hat{i} + asin(t)\hat{j}$, $0 \le t \le 2\pi$.

   (a) $\vec{F} = -x^2y\hat{i} + xy^2\hat{j}$ (b) $\vec{F} = y\hat{i}$

2. Use Green's theorem to find the counter-clockwise circulation and outward flux for the field $\vec{F} = (x + y)\hat{i} - (x^2 + y^2)\hat{j}$ and curve C which is a triangle bounded by $y = 0$, $x = 1$, and $y = x$.

3. Find the work done by $\vec{F} = (4x - 2y)\hat{i} + (2x - 4y)\hat{j}$ in moving a particle once counter-clockwise around the curve $C : (x - 2)^2 + (y - 2)^2 = 4$.

4. Apply Green's theorem to evaluate the following integral :

   (a) $\oint_c (3ydx + 2xdy)$ where C: The boundary of $0 \le x \le \pi, 0 \le y \le \sin x$

   (b) $\oint_c (2x + 3y^2)\, dx + (2xy + 3y)dy$ where C : Any simple closed curve in the plane for which Green's theorem holds.

5. Use Green's theorem area formula $A = \oint_C \frac{1}{2}(xdy - ydx)$ find the area of the region enclosed by the curve $\vec{r}(t) = t^2\hat{i} + ((t^3/3) - t)\hat{j}, -\sqrt{3} \le t \le \sqrt{3}$.

6. Show that if R is a region in the plane bounded by a piecewise-smooth simple close curve C, then

$$\text{Area of R} = \oint_c xdy = -\oint_c ydx$$

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7. Let $f(x, y) = \ln(x^2 + y^2)$ and let C be the circle $x^2 + y^2 = a^2$. Evaluate the flux integral

$$\oint_c \nabla f \cdot \vec{n}\, ds$$

8. Let A be the area and $\bar{x}$ the x-coordinate of the centroid of a region R that is bounded by a piecewise-smooth simple closed curve C in the xy-plane. Show that

$$\frac{1}{2}\oint x^2 dy = -\oint xydx = \frac{1}{3}\oint x^2dy - xydx = A\bar{x}$$

9. Evaluate surface integral $\iint \vec{F} \cdot \hat{n}dS$ where $\vec{F} = (x^2 + y^2 + z^2)(\hat{i} + \hat{j} + \hat{k})$, S is the surface of tetrahedron $x = 0$, $y = 0$, $z = 0$, $x + y + z = 2$ and $\hat{n}$ is the unit normal in the outward direction to the closed surface S.

10. Evaluate $\int \vec{A} \cdot d\vec{S}$, where $\vec{A} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ and S is the sphere $x^2 + y^2 + z^2 = a^2$.

11. Verify Divergence theorem for $\vec{F} = (x^2 - yz)\,\hat{i} + (y^2 - zx)\,\hat{j} + (z^2 - xy)\,\hat{k}$ taken over the rectangular parallelopiped $0 \leq x \leq z$, $0 \leq y \leq b$, $0 \leq z \leq c$.

12. Verify Green's theorem for $\oint_c (xy + y^2)\,dx + x^2 dy$, where C is the boundary of the closed region bounded by $y = x$ and $y = x^2$.

13. Apply Green's theorem to prove that the area enclosed by a plane curve is $\frac{1}{2} \oint xdy - ydx$. Hence find the area of the ellipse whose semi major and minor axes of lengths a and b.

14. Apply Stroke's theorem to find the value of $\int_c ydx + zdy + xdz$, where C is the curve of intersection of $x^2 + y^2 + z^2 = a^2$ and $x + z = a$.

15. Find the outward flux of $\vec{F} = y\hat{i} + xy\hat{j} - zk$ across the boundary of the region D, where D: The region inside the solid cylinder $x^2 + y^2 = 4$ between the plane $z = 0$ and the paraboloid $z = x^2 + y^2$.

16. Use the curl integral in Stoke's Theorem to find the circulation of the field $\vec{F} = y^2\hat{i} - y\hat{j} + 3z^2\hat{k}$ around the curve C in the indicated direction; C: The ellipse in which the plane $2x + 6y - 3z = 6$ meets the cylinder $x^2 + y^2 = 1$, counterclockwise as viewed from above.

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