Indian Institute of Technology Ropar
Department of Mathematics
MA101 - Calculus
First Semester of Academic Year 2025-26

Tutorial Sheet - 1

1. Let $x, y \in \mathbb{R}$. Then prove the following:
   (a) $max\{x, y\} = \frac{x+y+|x-y|}{2}$ and $min\{x, y\} = \frac{x+y-|x-y|}{2}$.
   (b) $0 \leq x < y \Rightarrow \frac{x}{1+x} < \frac{y}{1+y}$.
   (c) $\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}$.
2. For $x, y \in \mathbb{R}$, prove that
   (a) $|x + y| \geq ||x| - |y||$.
   (b) $|x - y| \geq ||x| - |y||$.
   (c) $|x - y| \leq |x| + |y|$.
3. Prove that $\sqrt{3}$ is an irrational number.
4. Prove that $1/\sqrt{1} + 1/\sqrt{2} + \cdots + 1/\sqrt{n} > \sqrt{n}$ for all $n \in \mathbb{N}$ and $n > 1$.
5. Let $a, b \in \mathbb{R}$, and suppose that for every $\epsilon > 0$ we have $a \leq b + \epsilon$. Show that $a \leq b$.
6. Let $a, b \in \mathbb{R}$. If $0 < a \leq b$ and the indicated square roots exist, then prove that
$$a \leq \frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}} \leq b.$$
7. Using Mathematical Induction prove that $\forall n \in \mathbb{N}, (1 + x)^n \geq 1 + nx$.
8. Show that if $a, b \in \mathbb{R}$, and $a \neq b$, then there exist ϵ-neighbourhoods U of a and V of b such that $U \cap V = \emptyset$.
9. Let $\epsilon > 0$ and $\delta > 0$ and $a \in \mathbb{R}$. Show that $V_\epsilon(a) \cap V_\delta(a)$ and $V_\epsilon(a) \cup V_\delta(a)$ are $\gamma-$ neighborhoods of a for appropriate values of γ.
10. Find all $x \in \mathbb{R}$ that satisfy both $|2x - 3| < 5$ and $|x + 1| > 2$ simultaneously.
11. Prove the following:
    (a) Every non-empty subset of $\mathbb{N}$ has a least member.
    (b) For every $x \in \mathbb{R}$, there exist $m, n \in \mathbb{Z}$ such that $m < x < n$.
    (c) For every $x \in \mathbb{R}$, there exists unique $n \in \mathbb{Z}$ such that $n \leq x \leq n + 1$, i.e., every real number lies between two consecutive integers.
12. Prove that there is no least positive real number.
13. Find $m \in \mathbb{N}$ such that $|\frac{2n}{n+3} - 2| < \frac{1}{5}, \forall n \geq m$.
14. Check whether the sequence $\{\frac{n^2+1}{2n+3}\}$ is bounded or not.
15. Prove or disprove that the following sequences are bounded:

(a) $\{\frac{2n-5}{2n+1}\}$.

(b) $\{1 + \frac{(-1)^n}{n}\}$.

(c) $\{1 + \frac{1}{5} + \frac{1}{5^2} + \ldots + \frac{1}{5^n}\}$.

(d) $\{n - n^2\}$.

16. Determine, whether the sequences are increasing or decreasing:

 (a) $\{\frac{n}{n^2+1}\}_{n\geq 1}$

 (b) $\{\frac{2^n 3^n}{5^{n+1}}\}_{n\geq 1}$

17. Prove that the sequence $\{x_n\}$, defined by $x_1 = \sqrt{a} > 0, x_{n+1} = \sqrt{a + x_n}$, is bounded above.

18. If $0 < a < b$ and $S_1 = a$, then the sequence $\{S_n\}$ is bounded and monotonic increasing, where $S_{n+1} = \sqrt{\frac{ab^2+S_n^2}{a+1}}$ for all $n \in \mathbb{N}$.

19. Prove that $[\frac{1}{2}(a+b)]^2 \leq \frac{1}{2}(a^2+b^2)$ for all $a, b \in \mathbb{R}$. Show that equality holds if and only if $a = b$.

20. Show that there exists a unique positive real number x such that $x^2 = 2$.

21. Show that if $a, b \in \mathbb{R}$ then

 (a) $\min\{a, b, c\} = \min\{\min\{a, b\}, c\}$.

 (b) $\min\{a, b\} = -\max\{-a, -b\}$.

---

* * * * * * ** ALL THE BEST * * * * * * **